Some Mathematics of Integral Value Transformations (IVTs) and it's Research scope

Ananya Roy,
Visiting Student,
Applied Statistics Unit, ISI Kolkata
Email: ananyaaroy1@gmail.com

Definition of IVT

$IVT^{p,k}{}_j : \mathbb{N}_0{}^K \rightarrow \mathbb{N}_0$

$IVT^{p,k}{}_j((n_1, n_2, \dots . n_k)=$

$(f_j(a_0{}^{n_1}, a_0{}^{n_2}, \dots, a_0{}^{n_k})\, f_j(a_1{}^{n_1}, a_1{}^{n_2}, \dots, a_1{}^{n_k}) \dots \dots f_j(a_{l-1}{}^{n_1}, a_{l-1}{}^{n_2}, \dots, a_{l-1}{}^{n_k}))_p = m$

where $n_1 = (a_0{}^{n_1} a_1{}^{n_1} \dots . a_{l-1}{}^{n_1})_p$, $n_2 = (a_0{}^{n_2} a_1{}^{n_2} \dots . a_{l-1}{}^{n_2})_p, \dots \dots n_k = (a_0{}^{n_k} a_1{}^{n_k} \dots . a_{l-1}{}^{n_k})_p$

f_j: $\{0, 1, 2, \dots, p-1\}^k \rightarrow \{0,1,2, \dots, p-1\}$.

m is the decimal conversion from the p adic number.

An example

For p=3, k=1. There are 3^{3^1} number of 1-dimension 3-variable functions two of which are given in the table below :

Variables	f_7	f_{16}
0	1	1
1	2	2
2	0	1

$x = 55 = (2001)_3$

$IVT^{3,1}{}_7(x) = (f_7(2)\, f_7(0) f_7(0) f_7(1))_3 = (0112)_3 = 14$

$IVT^{3,1}{}_{16}(x) = (f_{16}(2)\, f_{16}(0) f_{16}(0) f_{16}(1))_3 = (1112)_3 = 41$

Set theoretic form of IVT

Let us fix the domain of IVTs as $\mathbb{N}_0$ (k=1) and thus the above definition boils down to the following:

$$IVT^{p,1}{}_j\,(x) = \left(f_j(x_n)\,f_j(x_{n-1}) \dots \dots \dots f_j(x_1)\right)_p = m$$

where m is the decimal conversion from the p adic number, and $x = (x_n\, x_{n-1} \dots \dots x_1)_p$.

Let us denote the set of $IVT^{p,1}{}_j$ as

$$T^{p,1} = \left\{ IVT^{p,1}{}_j \;:\mathbb{N} \to \mathbb{N} \;\middle|\; \begin{array}{c} 0 \le j < p^p,\; IVT^{p,1}{}_j\,(x) = \left(f_j(x_n)\,f_j(x_{n-1}) \dots \dots \dots f_j(x_1)\right)_p = m \\ \text{where m is the decimal conversion from the p adic number} \\ \text{and } x = (x_n\, x_{n-1} \dots \dots x_1)_p \end{array} \right\}$$

We will now see some algebraic structures on the above function space in the next few slides

Algebraic Structures on $T^{p,1}$

Let us define two binary operations $\oplus$ and $\otimes$ as

$$(IVT^{p,1}{}_{j_1} \oplus IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \oplus_p f_{j_2}(x_n)\ \ f_{j_1}(x_{n-1}) \oplus_p f_{j_2}(x_{n-1}) \ldots \ldots \ldots f_{j_1}(x_1) \oplus_p f_{j_2}(x_1)\right)_p \text{ and}$$

$$(IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n)\ \ f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1}) \ldots \ldots \ldots f_{j_1}(x_1) \otimes_p f_{j_2}(x_1)\right)_p$$

where $x = (x_n\ x_{n-1} \ldots \ldots x_1)_p$ and $\oplus_p$ denotes addition modulo p and $\otimes_p$ denotes multiplication modulo p.

We see that $(T^{p,1}, \oplus)$ forms an abelian group and $(T^{p,1}, \otimes)$ forms a semi group and the two operations follow the distributive laws.

Thus, $(T^{p,1}, \oplus, \otimes)$ forms a Commutative Ring .

An attempt to create a field structure on $T^{p,1}$

We already know that ($T^{p,1}$, $\oplus$) is an abelian group where $\oplus$ is defined as

$$(\, IVT^{p,1}{}_{j_1} \oplus IVT^{p,1}{}_{j_2})(x) =$$

$$\left(f_{j_1}(x_n) \oplus_p f_{j_2}(x_n)\;\; f_{j_1}(x_{n-1}) \oplus_p f_{j_2}(x_{n-1}) \dots\dots\dots f_{j_1}(x_1) \oplus_p f_{j_2}(x_1)\right)_p$$

where $x = (x_n\; x_{n-1} \dots\dots x_1\,)_p$ and $IVT^{p,1}{}_{j_1}, IVT^{p,1}{}_{j_2} \in T^{p,1}$

Our task is now to define an operation $\otimes$ in such a way that ($T^{p,1}$, $\otimes$) forms an abelian group and follows the distributive laws.

Let us define an operation $\otimes$ as below:

$$(\, IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})\,(x) =$$

$$\left(\delta\left(f_{j_1}(x_n)\, x\, f_{j_2}(x_n)\right)\; \delta(f_{j_1}(x_{n-1})\, x\, f_{j_2}(x_{n-1})) \dots\dots\dots \delta(f_{j_1}(x_1)\, x\, f_{j_2}(x_1))\right)_p$$

where $\delta(x_i \; x \; x_j) = \begin{cases} x_i \otimes_p x_j\,, & \text{if } x_i \neq x_j \\ 1\,, & \text{if } x_i = x_j \end{cases}$ where $x_i, x_j \in \mathbb{F}_p$

($T^{p,1}$,$\otimes$) is an abelian group but it does not follow the distributive law as

$$IVT^{3,1}{}_{11} \otimes (IVT^{3,1}{}_{16} \oplus IVT^{3,1}{}_{19}) \neq IVT^{3,1}{}_{11} \otimes IVT^{3,1}{}_{16} \oplus IVT^{3,1}{}_{11} \otimes IVT^{3,1}{}_{19}\,.$$

Hence ($T^{p,1}$, $\oplus$, $\otimes$) fails to be a field. Thus, unfortunately our efforts in this direction have been unsuccessful so far.

Contd…

$(T^{p,1}{}_{\#}, \oplus, \wedge)$ forms a **Vector space over a field** $\mathbb{F}_p$ **(p is prime)** where ^ denotes scalar multiplication defined as $(c \wedge IVT^{p,1}{}_j)(x) = \left(c \otimes_p f_j(x_n) \quad c \otimes_p f_j(x_{n-1}) \ldots\ldots\ldots \; c \otimes_p f_j(x_1)\right)_p$

where $x = (x_n \; x_{n-1} \ldots\ldots x_1)_p$ and $\otimes_p$ denotes multiplication modulo p.

Remark: When p is prime, $T^{p,1}$ is a finite vector space (with p^p functions) with $\dim(T^{p,1}) = p$ over the finite field $\mathbb{F}_p$. When p is composite number, $(T^{p,1}, \oplus, \wedge)$ forms a module over $\mathbb{F}_p$ which is a commutative ring with unity under addition and multiplication modulo p. Moreover, it is a *free module* since it has a basis which will be shown in the next slide.

Basis functions for the vector space

The basis of $(T^{p,1}, \oplus, \wedge)$ is { $IVT^{p,1}{}_j$ such that $j=p^i$ where $i=0,1,2,\ldots p-1$ }.

Thus for any $IVT^{p,1}{}_j \in T^{p,1}$, $IVT^{p,1}{}_j = a_0 \wedge IVT^{p,1}{}_{p^0} + a_1 \wedge IVT^{p,1}{}_{p^1} + a_2 \wedge IVT^{p,1}{}_{p^2} \ldots\ldots\ldots\ldots + a_{p-1} \wedge IVT^{p,1}{}_{p^{p-1}}$

where $a_i \in \mathbb{F}_p$ and $j=\sum_{i=0}^{p-1} a_i\, p^i$.

Illustration:

Basis functions of $T^{3,1}$ are $IVT^{3,1}{}_1$, $IVT^{3,1}{}_3$, $IVT^{3,1}{}_9$

In $T^{3,1}$, we can write $IVT^{3,1}{}_{21} = a_0 \wedge IVT^{3,1}{}_{3^0} + a_1 \wedge IVT^{3,1}{}_{3^1} + a_2 \wedge IVT^{3,1}{}_{3^2}$ where $a_i \in \mathbb{F}_3$, $i=0,1,2$

and $21 = a_0 + 3a_1 + 9a_2$. So we must have $a_0 = 0, a_1 = 1, a_2 = 2$.

Thus we must have

$IVT^{3,1}{}_{21} = 0 \wedge IVT^{3,1}{}_{3^0} + 1 \wedge IVT^{3,1}{}_3 + 2 \wedge IVT^{3,1}{}_9$

Now,

- We have a way expressing any function in the p-adic function space in terms of the basis functions of the p-adic space.

- Next we would like to see the relation between functions in the p-adic and (p+1)-adic function spaces. To do this , we have devised a mechanism by defining a transformation between the bases sets of each space which can be extended to the whole space.

p-adic to (p+1)-adic basis functions

Let us define a function $T : B_p \cup \{ IVT^{p,1}{}_0 \} \rightarrow B_{p+1}$ as

$$T(IVT^{p,1}{}_j) = \begin{cases} IVT^{(p+1),1}{}_{(p+1)^p} & \text{if } \quad j = 0 \\ IVT^{(p+1),1}{}_{(p+1)^i} & \text{if } j = p^i \mid i = 0,1,\dots p-1 \end{cases}$$

B_p and B_{p+1} are the bases of $T^{p,1}$ and $T^{p+1,1}$ respectively and

$f^{(p+1),1}{}_j : \{0,1,2,\dots\dots p-1, p\} \rightarrow \{0,1,2,\dots\dots\dots p-1, p\}$ is defined as

$$f^{(p+1),1}{}_j(p) = \begin{cases} 1 & \text{if } \quad f^{(p+1),1}{}_j(i) = 0 \ \forall\, i = 0,1,2,\dots p-1 \\ 0 & \text{if } \ f^{(p+1),1}{}_j(i) = 1 \text{ for some } i = 0,1,2,\dots p-1. \end{cases}$$

Then T is an Isomorphism.

An Illustration for the basis functions

Through the Transformation , $T : B_2 \cup \{ IVT^{p,1}{}_0 \} \rightarrow B_3$

, we have

$$IVT^{2,1}{}_0 \leftrightarrow IVT^{3,1}{}_9$$

$$IVT^{2,1}{}_1 \leftrightarrow IVT^{3,1}{}_1$$

$$IVT^{2,1}{}_2 \leftrightarrow IVT^{3,1}{}_3$$

Extension of T leads to…

Through an extension of the transformation T, we get $IVT^{2,1}{}_3 \rightarrow IVT^{3,1}{}_4$.

We first express $IVT^{2,1}{}_3$ in terms of the basis functions and then apply the transformation T on it,

$$IVT^{2,1}{}_3 = IVT^{2,1}{}_1 + IVT^{2,1}{}_2$$

$$T(IVT^{2,1}{}_3) = T(IVT^{2,1}{}_1 + IVT^{2,1}{}_2)$$

$$= IVT^{3,1}{}_1 + IVT^{3,1}{}_3 = IVT^{3,1}{}_4$$

Some properties of IVT

There are **p** number of linear functions in $T^{p,1}$ since for any linear function $f_{\#}$, we must have $f_{\#}$ (0) =0 and $f_{\#}(1) = i$ for $i \in \{0, 1, 2, \ldots \ldots p-1\}$ for a p-adic system. Consequently we would have $f_{\#}(r) = r \otimes_p f_{\#}(1)$ where $0 \leq \# \leq (p-1)$.

There are **p**! number of bijective functions in $T^{p,1}$.

All the p^p functions are surjective but not all of them would be injective. For a p-adic function , there are p number of variables i.e, 0,1,2,………p-1 and an injective function say f_j would have to take one of these variables again to one of these variables and hence we would have p! number of possibilities of arrangements of these variables.

An Illustration

For p=3

Linear functions:

Variables	f_0	f_{21}	f_{15}
0	0	0	0
1	0	1	2
2	0	2	1

Bijective functions:

Variables	f_{21}	f_{15}	f_7	f_{19}	f_{11}	f_5
0	0	0	1	1	2	2
1	1	2	2	0	0	1
2	2	1	0	2	1	0

Thus, so far some of the algebraic properties of the space of Integral Value Transformations have been highlighted .

Now, we will explore some analytical notions like derivability of these functions.

Derivability of IVT's

- For any function, to introduce the notion of derivability at a point , we first talk of the neighborhood of a point .

- Since IVT's are discrete functions, the neighborhood around a point will be a discrete set. Let us see the difference in the concept of neighborhood in the continuous and discrete cases.

Neighborhood around a point x_0

In case neighbourhood of a point in a continuous domain, it's defined as below

$N(x_0, r) = \{x_0 \in X : d(x, x_0) < r\}$ where d denotes the metric defined on the set X.

Now, for $x_0 \in \mathbb{N}_0$, the neighbourhood of x_0 consisting of discrete points

boils down to the following

$$N(x_0, r) = \{x \in \mathbb{N}_0 : |x - x_0| < r\} =$$

$$\{x_0 - r + 1, \dots \dots \dots . x_0 + r - 1\}; r > 1, r \in \mathbb{N}_0$$

Next we define an operator on the space of functions for derivability at a point.

Differentiation

To define an operator, D for differentiation, the following fundamental rules should be satisfied.

1) Linearity: $D(IVT^{p,1}_{j_1} + IVT^{p,1}_{j_2}) = D(IVT^{p,1}_{j_1}) + D(IVT^{p,1}_{j_2})$

2) Homogeneity: $D(k \hat{} IVT^{p,1}_{j_1}) = k\, D(IVT^{p,1}_{j_1})$

3) Leibnitz Rule: $D(IVT^{p,1}_{j_1} \,.\, IVT^{p,1}_{j_2}) = IVT^{p,1}_{j_1} \,.\, D(IVT^{p,1}_{j_2}) + IVT^{p,1}_{j_2} \,.\, D(IVT^{p,1}_{j_1})$

Now we are in a position to define the operator D on $(T^{p,1}, \oplus, \wedge)$

We first define a left derivative for an $IVT^{p,1}{}_j$ at an arbitrary point $c \in \mathbb{N}_0$ as the following:

Let $LD(IVT^{p,1}{}_j)(c)$ denote the *left derivative* of the function $IVT^{p,1}{}_j$ at the point c defined as

$$LD(IVT^{p,1}{}_j)(c) = \min_{x:x\in N(c,\varepsilon)} \left\{ \frac{IVT^{p,1}{}_j(x) - IVT^{p,1}{}_j(c)}{(x-c)} \right\}$$

Let $RD(IVT^{p,1}{}_j)(c)$ denote the *right derivative* of the function $IVT^{p,1}{}_j$ at the point c defined as

$$RD(IVT^{p,1}{}_j)(c) = \max_{x:x\in N(c,\varepsilon)} \left\{ \frac{IVT^{p,1}{}_j(x) - IVT^{p,1}{}_j(c)}{(x-c)} \right\}.$$

If both the right and left derivative of the function $IVT^{p,1}{}_j$ exist at the point c and are equal, then we say that $IVT^{p,1}{}_j$ is differentiable at the point c and the derivative at c is equal to $D(IVT^{p,1}{}_j)(c)$.

We see that….

Linearity:

$$LD(\ IVT^{p,1}{}_{j_1} + IVT^{p,1}{}_{j_2})(c) = LD(\ IVT^{p,1}{}_{j_1})(c) + LD(IVT^{p,1}{}_{j_2})(c)$$

Leibnitz Rule:

$$LD(\ IVT^{p,1}{}_{j_1}.IVT^{p,1}{}_{j_2})(c)$$

$$= \min_{x:x\epsilon N(c,\varepsilon)}\{\ IVT^{p,1}{}_{j_1}(x)\ \}\ LD(\ IVT^{p,1}{}_{j_2})(c) + IVT^{p,1}{}_{j_2}(c).LD(\ IVT^{p,1}{}_{j_1})(c)$$

Similarly, the above holds for the right hand derivative and are equal if the function is derivable.

However, Homogeneity does not hold from the fact that

$$LD(\ k\text{^}\ IVT^{p,1}{}_{j})(c) = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{k\text{^}\ IVT^{p,1}{}_{j}(x) - k\text{^}\ IVT^{p,1}{}_{j}(c)}{(x-c)} \right\}$$

$$= \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{k\text{^}\ IVT^{p,1}{}_{j}(x) - k\text{^}\ IVT^{p,1}{}_{j}(c)}{(x-c)} \right\}$$

$$= LD(IVT^{p,1}{}_{j_1})(c) \text{ for some } j_1.$$

An illustration

Consider, $IVT^{2,1}{}_{2}(x) = x$

Let c be any arbitrary natural number,

$$LD(IVT^{2,1}{}_{2})(c) = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_{2}(x) - IVT^{2,1}{}_{2}(c)}{(x-c)} \right\}$$

$$= \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{x - c}{(x-c)} \right\} = 1$$

$$RD(IVT^{2,1}{}_{2})(c) = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_{2}(x) - IVT^{2,1}{}_{2}(c)}{(x-c)} \right\}$$

$$= \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{x - c}{(x-c)} \right\} = 1$$

Therefore $IVT^{2,1}{}_{2}$ is *everywhere differentiable* as both $LD(IVT^{2,1}{}_{2})(c)$ and $RD(IVT^{2,1}{}_{2})(c)$ exist and are equal and the derivative is 1.

Another example..

$IVT^{2,1}{}_1(x) = (2^s - 1) - x$ where x has a s-bit representation

$$LD(IVT^{2,1}{}_1)(c) = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_1(x) - IVT^{2,1}{}_1(c)}{(x-c)} \right\}$$

$$= \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{S_1}-1) - x - [(2^{S_2}-1) - c]}{(x-c)} \right\} = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{S_1} - 2^{S_2})}{(x-c)} - 1 \right\} \rightarrow -1$$

$$RD(IVT^{2,1}{}_1)(c) = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_1(x) - IVT^{2,1}{}_1(c)}{(x-c)} \right\}$$

$$= \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{S_1}-1) - x - [(2^{S_2}-1) - c]}{(x-c)} \right\} = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{S_1} - 2^{S_2})}{(x-c)} - 1 \right\} \rightarrow +\infty$$

As $LD(IVT^{2,1}{}_1)(c)$ and $RD(IVT^{2,1}{}_1)(c)$ are not equal therefore $IVT^{2,1}{}_1$ is *nowhere differentiable*.

Collatz like behavior

Collatz Conjecture:

If T : $\mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that T(n) = $\begin{cases} \frac{n}{2}, & if\ n\ is\ even \\ 3n+1, & if\ n\ is\ odd \end{cases}$

Then ∃ m ∈ $\mathbb{N}_0$ such that T^m(n)=1 for any choice of n.

We call T as a Collatz function.

$\mathrm{IVT}^{p,1}{}_j$ is called a Collatz like function if ∃ m ∈ $\mathbb{N}_0$ such that $(\mathrm{IVT}^{p,1}{}_j)^m$(n)=0 for any choice of n.

What now?

- Now, we have the basic algebraic structures to work with and some basic analytical notions which sets the stage for further research.

- We have plenty of IVT's in any p-adic system, some of which have a Collatz like behavior. For the rest, we need to understand the dynamism of these functions for which we introduce a dynamical system in the next section.

- There are many ways of looking at a dynamical system and one of the rich ways is to look through topological dynamics.

Discrete Dynamical System

Let G be a semi-group with respect to f and U be any space. Let

T : G X U $\rightarrow U$ defined as

T(g,x) = T_g(x) such that $f(T_g, T_h) = T_{f(g,h)}$

where $T_g : U \rightarrow U$

Then (G, T) is called a dynamical system.

If G = $\mathbb{N}_0$ or Z, then it's called a discrete dynamical system.

Discrete Dynamical System of IVT

Let us define

$T : \mathbb{N}_0 \; X \; \mathbb{N}_0 \rightarrow \mathbb{N}_0$ as

$T(n, x) = T_n(x)$

Where $T_n(x) = (IVT^{p,1}{}_{j})^{n}(x) = \underbrace{[(IVT^{p,1}{}_{j})\,(IVT^{p,1}{}_{j}) \ldots \ldots (IVT^{p,1}{}_{j})]}_{n \; times}\; (x)$

$(\mathbb{N}_0, T)$ is a discrete dynamical system.

$T(n, x)$ is the evolution function and n is the evolution parameter of the dynamical system.

X_0 is the initial state and $\mathbb{N}_0$ is the state/phase space. Corresponding to each p and j, we get a different dynamical system.

Orbits

The orbit of x is a set of points denoted by

$$\gamma(x) = \{x_n \,|\, n \in \mathbb{N}_0, x = x_0, x_{n+1} = (IVT^{p,1}{}_j)(x_n)\}$$

$$= \{x_0, (IVT^{p,1}{}_j)(x_0), (IVT^{p,1}{}_j)^2(x_0), (IVT^{p,1}{}_j)^3(x_0), \ldots\ldots\ldots\}$$

We know $IVT^{2,1}{}_1$ is a Collatz like function so its orbit around any point should be finite and containing 0. It's orbit around x_0 is given by

$\gamma(x_0) = \{x_0, (IVT^{2,1}{}_1)(x_0), (IVT^{2,1}{}_1)^2(x_0), (IVT^{2,1}{}_1)^3(x_0), \ldots\ldots\ldots\}$

$\gamma(0) = \{0,1\}$, $\gamma(1) = \{0,1\}$, $\gamma(2) = \{2,1,0\}$, $\gamma(4) = \{4,3,1,0\}$ and so on

It's worth noting that the orbits of the Merseene numbers are 3-point sets which consists of the number itself and 0 and 1 that is $\gamma(3) = \{3,1,0\}$, $\gamma(7) = \{7,1,0\}$, $\gamma(15) = \{15,1,0\}$,...

The orbit of any steady state equilibrium/fixed point will be the point itself.

Classification of orbits

We can classify the orbits into 3 broad categories namely:

- steady state equilibria / fixed points
- periodic points
- non-periodic points

This classification gives us a better understanding of the dynamism of the IVTs.

Factors and Topological Conjugacy

If (X, f) and (Y, g) are two dynamical systems , then (Y, g) is called a factor of (X, f) if there exists a continuous surjection $h: X \to Y$ such that $h \cdot f = g \cdot h$.

If the function h is also a homeomorphism, then the two systems are called topologically conjugate.

Topological Conjugacy of IVT induced DS

For any $IVT^{p,1}{}_{j_1}$ and $IVT^{p,1}{}_{j_2}$, the dynamical systems ($\mathbb{N}_0$, $IVT^{p,1}{}_{j_1} \cdot IVT^{p,1}{}_{j_2}$) and ($\mathbb{N}_0$, $IVT^{p,1}{}_{j_2} \cdot IVT^{p,1}{}_{j_1}$) will be factors of each other since

$IVT^{p,1}{}_{j_1} \cdot (IVT^{p,1}{}_{j_2} \cdot IVT^{p,1}{}_{j_1}) = (IVT^{p,1}{}_{j_1} \cdot IVT^{p,1}{}_{j_2}) \cdot IVT^{p,1}{}_{j_1}$ and

$IVT^{p,1}{}_{j_2} \cdot (IVT^{p,1}{}_{j_1} \cdot IVT^{p,1}{}_{j_2}) = (IVT^{p,1}{}_{2} \cdot IVT^{p,1}{}_{j_1}) \cdot IVT^{p,1}{}_{j_2}$ where $IVT^{p,1}{}_{j_1}$ and $IVT^{p,1}{}_{j_2}$ are continuous surjections.

($\mathbb{N}_0$, $IVT^{3,1}{}_{13}$) is a factor of ($\mathbb{N}_0$, $IVT^{3,1}{}_{18}$) since

$IVT^{3,1}{}_{16} \cdot IVT^{3,1}{}_{18} = IVT^{3,1}{}_{13} \cdot IVT^{3,1}{}_{16}$ $(= IVT^{3,1}{}_{13})$ where $IVT^{3,1}{}_{16}$ is a continuous surjection.

Now let us define a mapping h: $\mathbb{N}_0 \rightarrow \mathbb{N}_0$ as

$h(x)=h((x_n\ x_{n-1} \ldots \ldots x_1\ )_p) = ((x_n \oplus_p 1)\ (x_{n-1} \oplus_p 1) \ldots \ldots (x_1 \oplus_p 1))_p$.

Clearly, h is a homeomorphism. ($\mathbb{N}_0$, $IVT^{3,1}{}_{16}$) and ($\mathbb{N}_0$, $IVT^{3,1}{}_{8}$) are topologically conjugate dynamical systems as $h \cdot IVT^{3,1}{}_{16} = IVT^{3,1}{}_{8} \cdot h$. Thus these two systems show the same dynamical behaviour.

Another direction

The dynamical system defined before is a non-linear system hence we can do the stability analysis of the fixed points in order to see how chaotic the dynamical system is. Let us discuss this tritely.

Let x_0 be the initial condition $T_0(x_0) = (IVT^{p,1}{}_j)^0(x_0) = x_0$

$x_1 = T_1(x_0) = (IVT^{p,1}{}_j)^1(x_0)$, $x_2 = T_2(x_0) = (IVT^{p,1}{}_j)^2(x_0)$

$x_3 = T_3(x_0) = (IVT^{p,1}{}_j)^3(x_0)$,.....

$\{x_n\}$ gives us the trajectory of the non-linear system and the iterative scheme is given by

$x_n = IVT^{p,1}{}_j(x_{n-1})$.

Stability Analysis of steady state equilibrium of discrete dynamical systems is based on some propositions and/or explicit solution of the non linear, autonomous (the parameters/coefficients a and b in the difference equation $x_n = ax_{n-1} + b$ are independent of time), one-dimensional dynamical systems after reducing the non-linear system to a linear system.

What do we mean by stability?

A linear system is called locally stable if for a small perturbation to the system, it converges asymptotically to the original equilibrium. A linear system is called globally stable if irrespective of the extent of perturbation, it converges asymptotically to the original equilibrium. Mathematically, the definition is as follows:

A steady state equilibrium $\bar{x}$, of the linear difference equation $x_n = ax_{n-1} + b$ is called

- globally(asymptotically) stable if $\lim_{n\to\infty} x_n = \bar{x}$ for all $x_0 \in \mathbb{N}_0$.
- locally (asymptotically) stable if there exists r >0 such that $\lim_{n\to\infty} x_n = \bar{x}$ for all $x_0 \in N(\bar{x}, r)$, a neighbourhood of the point $\bar{x}$.

Conditions for local stability

Now, for the non-linear system $x_n = IVT^{p,1}{}_j(x_{n-1})$ ......(1)

We reduce the non-linear system to a linear system around the steady state equilibrium $\bar{x}$.

The linearized system around the fixed point is

$x_{n+1} = ax_{n-1} + b$ ....(2)

where $a = DIVT^{p,1}{}_j(\bar{x})$ and $b = [IVT^{p,1}{}_j(\bar{x}) - \bar{x}DIVT^{p,1}{}_j(\bar{x})]$

The linearized system (2) is globally stable iff $|a| = |DIVT^{p,1}{}_j(\bar{x})| < 1$ and convergence is monotonic if $0<a<1$ and oscillatory if $-1<a<0$

Thus, the dynamical system (1) is locally stable around the steady state equilibrium/fixed point $\bar{x}$ iff $|DIVT^{p,1}{}_j(\bar{x})| < 1$.

Conditions for global stability

Banach's Fixed Point Theorem:

If (X,d) is a complete metric space and $f:X \rightarrow X$ is a contraction mapping , then f will have a unique fixed point.

where $f:X \rightarrow X$ is a contraction mapping means $d(f(x), f(y)) \leq \lambda\, d(x, y)$ for all $x, y \in X$ and for some $\lambda \in (0, 1)$.

For deriving a condition for the global stability of a unique steady state equilibrium, we will invoke Banach's fixed point theorem. To do this, we first define a metric d on $\mathbb{N}_0$ such that $(\mathbb{N}_0 , d)$ is a complete metric space and find all the contraction mappings for some p and j in a p-adic system. Now we apply the fixed point theorem to arrive at a unique fixed point(which ensures the global stability of the non-linear system).

References

- *Collatz Function like Integral Value Transformations*, Sk. S. Hassan et al, (2010): ALEXANDRIA JOURNAL OF MATHEMATICS
- *Topological Dynamics of 2D Cellular Automata*: Mathieu Sablik et al, http://arxiv.org/abs/0709.4565
- *STUDIES IN TOPOLOGICAL DYNAMICS WITH EMPHASIS ON CELLULAR AUTOMATA*, PhD thesis of T K S Moothathu.
- *Number Theory and Dynamical Systems* , Jeffrey C. Lagarias, *The Unreasonable Effectiveness of Number Theory* , (S. A. Burr, Ed.), Proc. Symp. Applied Math. No. 46, AMS: Providence 1992, pp. 35-72.

References

- Pabitra Pal Choudhury et al *Theory of Rule 6 and Its Application to Round Robin Tournament,* International Journal of Computational Cognition, vol. 8, no. 3, September 2010, pp 37-41
- Sk. S. Hassan et al, *Act of CVT and EVT in The Formation of Number Theoretic Fractals*, International Journal of Computational Cognition Vol. 9, No. 1, pp: 1-6.
- Pabitra Pal Choudhury et al, *Theory of Carry Value Transformation (CVT) and it's Application in Fractal formation,* GJCST (2010) Volume 10 Issue 14: 89-98.

Acknowledgement

- Prof. Pabitra Pal Choudhury, ASU, ISI, Kolkata.
- Sk. Sarif Hassan, ASU, ISI Kolkata.

THANK YOU!!